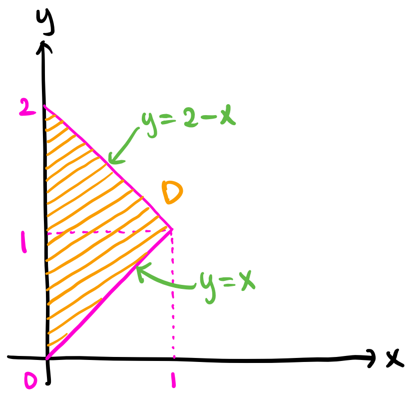
1. (10 points) Evaluate the double integral

$$\int\int_D xy\,dA,$$

where D is the triangular region with vertices at $(0,0)$, $(0,2)$, and $(1,1)$.



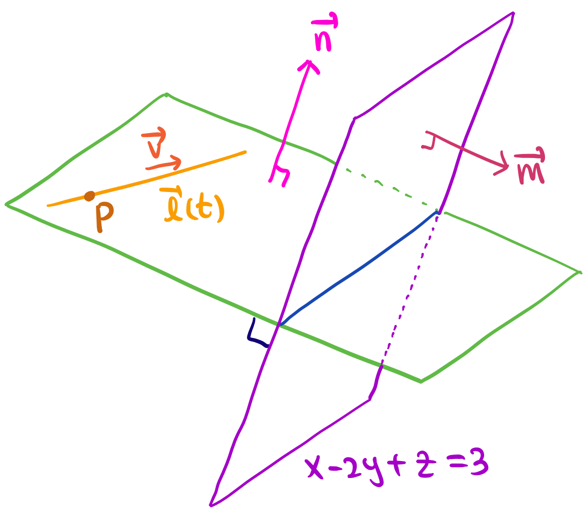
D is given by $0 \le x \le 1,\ x \le y \le 2-x$.

$$\iint_D xy\,dA = \int_0^1 \int_x^{2-x} xy\,dy\,dx = \int_0^1 \frac{xy^2}{2}\Big|_{y=x}^{y=2-x} dx$$

$$= \frac{1}{2}\int_0^1 (2-x)^2 x - x^3\,dx = \frac{1}{2}\int_0^1 4x - 4x^2\,dx$$

$$= \left(x^2 - \frac{2}{3}x^3\right)\Big|_{x=0}^{x=1} = \boxed{\frac{1}{3}}$$

2. (10 points) Two planes are said to be orthogonal if their normal vectors are orthogonal. Find the equation of a plane that is orthogonal to the plane $x - 2y + z = 3$ and contains the line with parametric equation $(x, y, z) = (1 + 2t, 2 - t, -1 + 2t)$.



The plane $x-2y+z=3$ has a normal vector $\vec{m} = (1,-2,1)$.

The line $\vec{\ell}(t) = (1+2t, 2-t, -1+2t)$ has a direction vector $\vec{v} = (2,-1,2)$.

A normal vector $\vec{n}$ of the desired plane is orthogonal to both $\vec{m}$ and $\vec{v}$.

$\Rightarrow \vec{n} = \vec{m} \times \vec{v} = (1,-2,1) \times (2,-1,2) = (-3,0,3)$

Moreover, the desired plane contains the point $P = \vec{\ell}(0) = (1,2,-1)$.

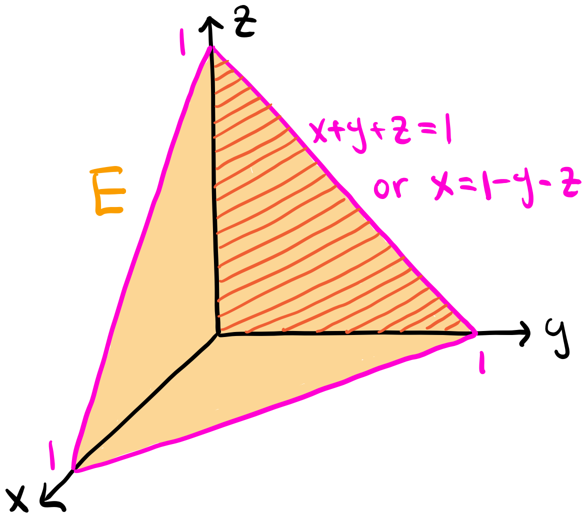
$\Rightarrow$ The desired plane is given by $\boxed{-3(x-1) + 0\cdot(y-2) + 3(z+1) = 0}$

__Note__ You can choose a different point on the line $\vec{\ell}(t)$.

3. (10 points) Let $\mathbf{F} = xz\mathbf{i} + y\mathbf{j} + x\mathbf{k}$. Find the flux

$$\int \mathbf{F} \cdot \mathbf{n}\, dS$$

out of the surface of the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Here $\mathbf{n}$ is the outward normal as usual.



S : the surface of the tetrahedron with outward orientation.

E : the solid bounded by S.

$\Rightarrow \partial E = S$ is oriented outward.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\text{div. thm}}{=} \iiint_E \operatorname{div}(\vec{F})\, dV. \quad (*)$$

E is bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$.

* As a general tip, the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ has x-intercept a, y-intercept b, z-intercept c.

The shadow on the yz-plane : $0 \le z \le 1$, $0 \le y \le 1-z$.

For each point on the shadow: $0 \le x \le 1-y-z$.

$\Rightarrow E$ is given by $0 \le z \le 1$, $0 \le y \le 1-z$, $0 \le x \le 1-y-z$.

$\vec{F} = (xz, y, x) \Rightarrow \operatorname{div}(\vec{F}) = z+1$

$$\iint_S \vec{F}\cdot d\vec{S} \underset{(*)}{=} \iiint_E z+1\, dV = \int_0^1\int_0^{1-z}\int_0^{1-y-z} z+1\, dx\,dy\,dz = \int_0^1\int_0^{1-z} (1-y-z)(z+1)\, dy\,dx$$

$$= \int_0^1 \left((1-z)y - \frac{y^2}{2}\right)(z+1)\Big|_{y=0}^{y=1-z} dz = \int_0^1 \frac{(1-z)^2(z+1)}{2}\, dz$$

$$\underset{u=z-1}{=} \int_{-1}^0 \frac{u^2(u+2)}{2}\, du = \left(\frac{u^4}{8}+\frac{u^3}{3}\right)\Big|_{u=-1}^{u=0} = \boxed{\frac{5}{24}}$$

4. Below $\mathbf{F}(x,y,z)$ is a vector field and $f(x,y,z)$ is scalar valued.

(a) (5 points) Find f such that $\mathbf{F} = \nabla f$ for $\mathbf{F} = z\cos y\mathbf{i} - xz\sin y\mathbf{j} + x\cos y\mathbf{k}$.

* The answer for this part given on the archive has an error.

$\vec{F} = (z\cos(y), -xz\sin(y), x\cos(y))$

$\Rightarrow P = z\cos(y),\ Q = -xz\sin(y),\ R = x\cos(y)$

$\text{curl}(\vec{F}) = (0,0,0) \Rightarrow \vec{F}$ is conservative.

$\vec{F} = \nabla f \Rightarrow P = f_x,\ Q = f_y,\ R = f_z$

$\int P\,dx = \int z\cos(y)\,dx = \boxed{xz\cos(y)}$

$\int Q\,dy = \int -xz\sin(y)\,dy = \boxed{xz\cos(y)}$

$\int R\,dz = \int x\cos(y)\,dz = \boxed{xz\cos(y)}$

$\Rightarrow f(x,y,z) = \boxed{xz\cos(y)}$

(To find $f(x,y,z)$, collect all terms without duplicates)

Note You get a different potential function by adding a constant.

(b) (5 points) Verify that there is no f with $\mathbf{F} = \nabla f$ for $\mathbf{F} = z\cos y\mathbf{i} + xz\sin y\mathbf{j} + x\cos y\mathbf{k}$.

$\vec{F} = (z\cos(y), xz\sin(y), x\cos(y))$

$\Rightarrow \text{curl}(\vec{F}) = (-2x\sin(y), 0, 2z\sin(y)) \neq (0,0,0)$

$\Rightarrow \vec{F}$ is not conservative

$\Rightarrow \vec{F}$ has $\boxed{\text{no potential functions}}$

5. Consider the solid sphere $x^2 + y^2 + z^2 \leq 1$.

(a) (6 points) Assume that the density (mass per unit volume) is equal to $z + 1$. Find the z-coordinate of the center of mass of the sphere.

E: the solid ball $x^2+y^2+z^2 \leq 1$ with density $\rho(x,y,z) = z+1$.

$$m = \iiint_E \rho(x,y,z)\,dV = \iiint_E z+1\,dV = \iiint_E z\,dV + \iiint_E 1\,dV$$

$\iiint_E z\,dV = 0$ by symmetry (*)

(the integrand z is odd with respect to z
the solid E is symmetric about the xy-plane)

$$\Rightarrow m = \iiint_E 1\,dV = \mathrm{vol}(E) = \underbrace{\frac{4}{3}\pi \cdot 1^3}_{\text{volume of sphere}} = \frac{4\pi}{3}$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV = \frac{3}{4\pi}\iiint_E z^2+z\,dV$$

$$= \frac{3}{4\pi}\iiint_E z^2\,dV + \frac{3}{4\pi}\iiint_E z\,dV \underset{(*)}{=} \frac{3}{4\pi}\iiint_E z^2\,dV$$

E: $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$, $0 \leq \rho \leq 1$ in spherical coordinates.

$$\Rightarrow \bar{z} = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\cos^2\varphi \cdot \underbrace{\rho^2\sin\varphi}_{\text{Jacobian}}\,d\rho\,d\varphi\,d\theta$$

$$= \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}\cos^2\varphi\sin\varphi\Big|_{\rho=0}^{\rho=1}\,d\varphi\,d\theta = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1}{5}\cos^2\varphi\sin\varphi\,d\varphi\,d\theta$$

$$\underset{u=-\cos\varphi}{=} \frac{3}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\frac{u^2}{5}\,du\,d\theta = \frac{3}{4\pi}\int_0^{2\pi}\frac{u^3}{15}\Big|_{u=-1}^{u=1}\,d\theta = \frac{3}{4\pi}\int_0^{2\pi}\frac{2}{15}\,d\theta = \boxed{\frac{1}{5}}$$

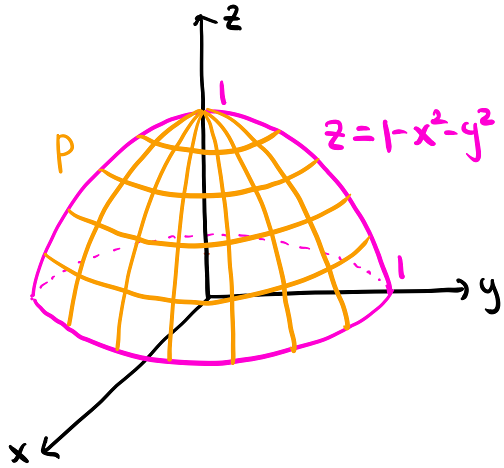
(b) (4 points) Assume that the density is equal to $x + 1$. Find the x-coordinate of the center of mass of the sphere.

Since the solid E remains the same upon swapping the x-axis and the z-axis, the answer for part (b) should be equal to the answer for part (a).

$$\Rightarrow \bar{x} = \boxed{\frac{1}{5}}$$

6. Consider the paraboloid surface P given by $z = 1 - (x^2 + y^2)$, $0 \le x^2 + y^2 \le 1$.

(a) (5 points) Find the area of the surface P.



The surface P is parametrized by

$\vec{r}(x,y) = (x, y, 1-x^2-y^2)$.

The domain D is given by $x^2+y^2 \le 1$.

$\vec{r}_x = (1, 0, -2x)$, $\vec{r}_y = (0, 1, -2y)$

$\Rightarrow \vec{r}_x \times \vec{r}_y = (2x, 2y, 1)$

$$\text{Area}(P) = \iint_P 1\,dS = \iint_D |\vec{r}_x \times \vec{r}_y|\,dA = \iint_D \sqrt{4x^2+4y^2+1}\,dA$$

In polar coordinates, D is given by $0 \le \theta \le 2\pi$, $0 \le r \le 1$.

$$\Rightarrow \text{Area}(P) = \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\cdot \overset{\text{Jacobian}}{r}\,dr\,d\theta \underset{u=4r^2+1}{=} \int_0^{2\pi}\int_1^5 u^{1/2}\cdot\frac{1}{8}\,du\,d\theta$$

$$= \int_0^{2\pi} \frac{u^{3/2}}{12}\Big|_{u=1}^{u=5} d\theta = \frac{1}{12}\int_0^{2\pi} 5^{3/2} - 1\,d\theta = \boxed{\frac{\pi}{6}(5^{3/2}-1)}$$

(b) (5 points) Evaluate the surface integral

$$\int\!\!\int_P \mathbf{F}\cdot\mathbf{n}\,dS,$$

where P is the same paraboloid surface and $\mathbf{F} = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j} + z\mathbf{k}$. As usual, $\mathbf{n}$ is the unit normal to the surface (in the upward direction) and dS is the area element. The disc at the bottom is *not* included in the surface.

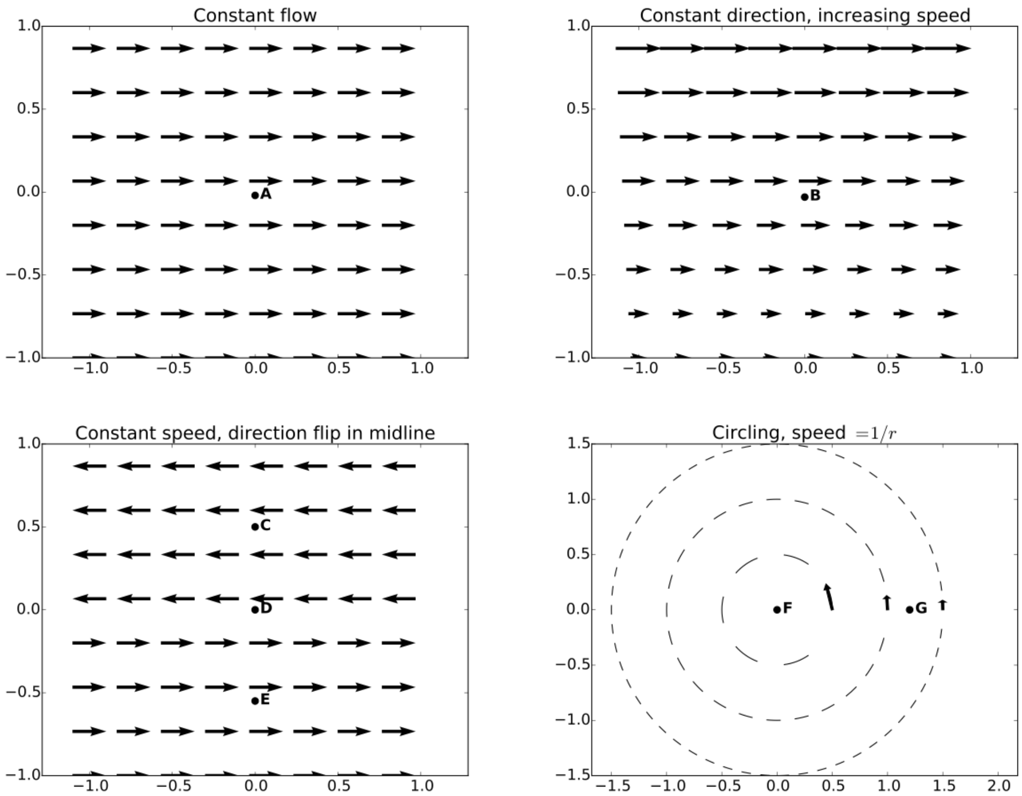
$\vec{F} = (\frac{x}{2}, \frac{y}{2}, z) \Rightarrow \vec{F}(\vec{r}(x,y)) = (\frac{x}{2}, \frac{y}{2}, 1-x^2-y^2)$

$\vec{r}_x \times \vec{r}_y = (2x, 2y, \underset{\vee\, 0}{1})$: oriented upward

$\vec{F}(\vec{r}(x,y))\cdot(\vec{r}_x \times \vec{r}_y) = (\frac{x}{2}, \frac{y}{2}, 1-x^2-y^2)\cdot(2x, 2y, 1) = 1$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x,y))\cdot(\vec{r}_x\times\vec{r}_y)\,dA = \iint_D 1\,dA = \text{Area}(D) = \underbrace{\pi\cdot 1^2}_{\text{Area of disk}} = \boxed{\pi}$$

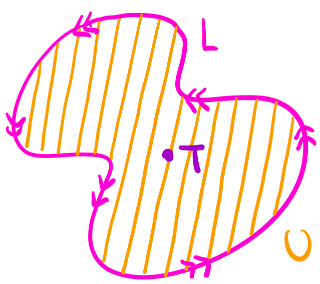
7. (10 points) The following four plots show vector fields or flows. It is assumed that the vector field has no z-component and that the flow is the same in all planes parallel to the x-y plane. Therefore the only component of the curl that can be nonzero is the z-component. The z-axis is perpendicular to the plane of the paper and pointing towards the sky.
For each of the points A to G, figure out if the z-component of the curl is negative, zero, or positive. Read the titles of each plot carefully. It is important that D is located exactly on the midline, where the flow flips direction. Likewise, it is significant that F is the center. Hint: consider loops of an appropriate shape and think in terms of Stokes' theorem.



The vector field takes the form $\vec{F} = (P, Q, 0)$ (no z-component)

$\Rightarrow$ The z-component of curl$(\vec{F})$ is $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

To find the sign of $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ at a point T, we draw a small loop L around T, oriented counterclockwise.



U : the domain enclosed by L

$\Rightarrow \partial U = L$ is positively oriented.

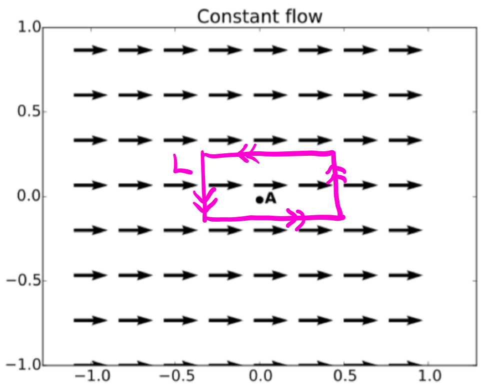
$$\int_L \vec{F}\cdot d\vec{r} = \iint_U \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

Green's thm

$\Rightarrow$ The sign of $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is equal to the sign of $\int_L \vec{F}\cdot d\vec{r}$

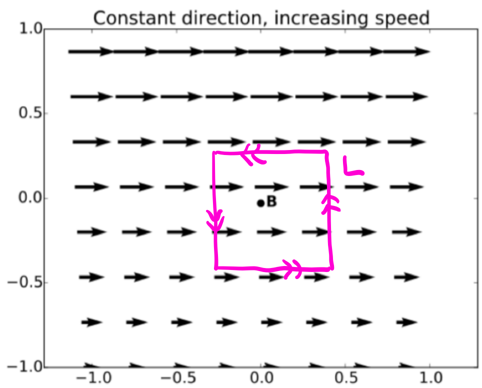
The sign of the line integral $\int_L \vec{F}\cdot d\vec{r}$ is given as follows :

- $\int_L \vec{F}\cdot d\vec{r}$ is positive if L and $\vec{F}$ are in the same direction
- $\int_L \vec{F}\cdot d\vec{r}$ is negative if L and $\vec{F}$ are in the opposite direction
- $\int_L \vec{F}\cdot d\vec{r}$ is zero if L and $\vec{F}$ are in the perpendicular direction.



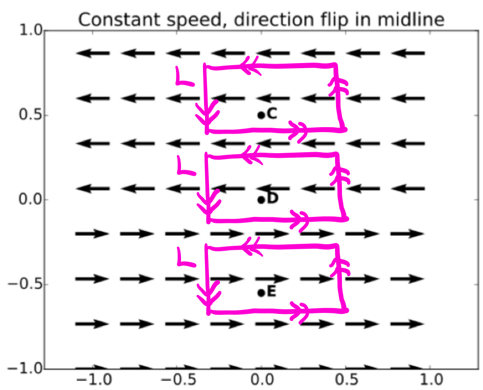
$\int_L \vec{F}\cdot d\vec{r} = 0$ at A

left / right sides : zero
top side : negative
bottom side : positive
equal magnitude



$\int_L \vec{F}\cdot d\vec{r} < 0$ at B

left / right sides : zero
top side : negative – larger magnitude
bottom side : positive – smaller magnitude

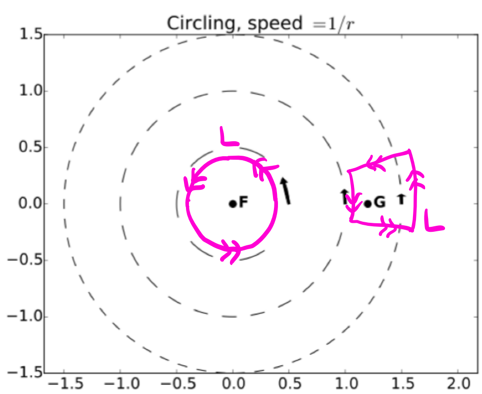


$\int_L \vec{F}\cdot d\vec{r} = 0$ at C and E

(by the same reson for A)

$\int_L \vec{F}\cdot d\vec{r} > 0$ at D

left / right sides : zero
top / bottom sides : positive



$\int_L \vec{F}\cdot d\vec{r} > 0$ at F.

$\int_L \vec{F}\cdot d\vec{r} = 0$ at G

top / bottom sides : zero
left side : negative
right side : positive
equal magnitude

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is $\boxed{\text{zero at A, C, E, G, positive at D, F, negative at B.}}$

8. Consider the line integral

$$\int_C \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy,$$

where C is assumed to be a simple closed curve with positive orientation.

(a) (3 points) Calculate the line integral for the curve $x^2 + y^2 = 1$.

This problem considers the integral of the vortex field

$\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

The curve C is a circle centered at the origin, oriented counterclockwise.

$\Rightarrow \int_C \vec{V} \cdot d\vec{r} = \boxed{2\pi}$ (see Fact 3(2) in the Final exam facts note)

(b) (3 points) Evaluate and simplify

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right).$$

$P = -\frac{y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(see Fact 3(1) in the Final exam facts note)

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \boxed{0}$

(c) (4 points) Calculate the line integral for the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$.

At $(0,0)$: $\frac{x^2}{16} + \frac{y^2}{25} = \frac{0^2}{16} + \frac{0^2}{25} = 0 < 1$

$\Rightarrow$ The curve C encloses the origin with counterclockwise orientation.

$\Rightarrow \int_C \vec{V} \cdot d\vec{r} = \boxed{2\pi}$ (see Fact 3(4) in the Final exam facts note)

9. The plane $3x + 6y + 2z = 7$ slices the spherical surface $x^2 + y^2 + z^2 = 4$ into two parts.

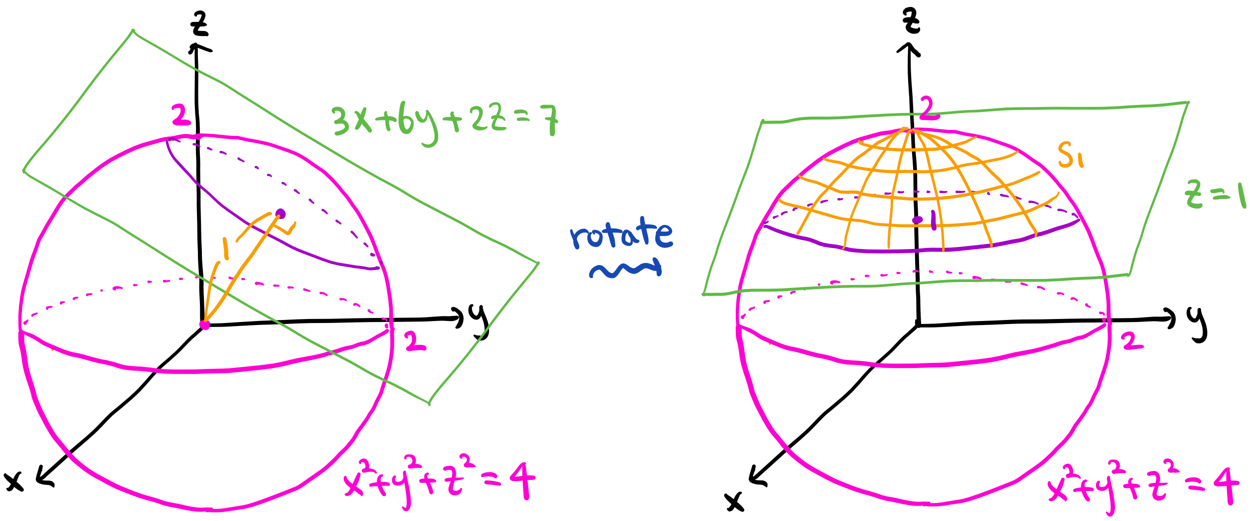
(a) (2 points) Find the distance of the plane from the center of the sphere.

The center of the sphere $x^2+y^2+z^2=4$ is the origin.

The distance from $(0,0,0)$ to the plane $3x+6y+2z-7=0$ is

$$\frac{|3\cdot 0+6\cdot 0+2\cdot 0-7|}{\sqrt{3^2+6^2+2^2}} = \boxed{1}$$

(b) (8 points) Find the surface area of each part of the sphere.



We can rotate the plane $3x+6y+2z=7$ to the plane $z=1$.

S_1 : the surface above the plane $z=1$

S_2 : the surface below the plane $z=1$

$x^2+y^2+z^2=4 \Rightarrow z=\sqrt{4-x^2-y^2}$ on S_1 $(z\geq 0)$

S_1 is parametrized by $\vec{r}(x,y) = (x, y, \sqrt{4-x^2-y^2})$

On the intersection : $z=1 \Rightarrow x^2+y^2 = 4-z^2=3$

$\Rightarrow$ The domain D is given by $x^2+y^2\leq 3$.

$\Rightarrow \text{Area}(S_1) = \iint_{S_1} 1\,dS = \iint_D |\vec{r}_x\times\vec{r}_y|\,dA$

$\vec{r}_x = \left(1, 0, -\frac{x}{\sqrt{4-x^2-y^2}}\right)$, $\vec{r}_y = \left(0, 1, -\frac{y}{\sqrt{4-x^2-y^2}}\right)$

$\Rightarrow \vec{r}_x\times\vec{r}_y = \left(\frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1\right)$

$\Rightarrow |\vec{r}_x\times\vec{r}_y| = \sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1} = \frac{2}{\sqrt{4-x^2-y^2}}$

$$\text{Area}(S_1) = \iint_D \frac{2}{\sqrt{4-x^2-y^2}}\, dA$$

In polar coordinates, D is given by $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{3}$.

Jacobian

$$\Rightarrow \text{Area}(S_1) = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2}{\sqrt{4-r^2}} \cdot r\, dr\, d\theta \underset{u=4-r^2}{=} \int_0^{2\pi}\int_4^1 -\frac{1}{u^{1/2}}\, du\, d\theta$$

$$= \int_0^{2\pi} -2u^{1/2}\Big|_{u=4}^{u=1} d\theta = \int_0^{2\pi} 2\, d\theta = 4\pi.$$

$$\text{Area}(S_1) + \text{Area}(S_2) = \underbrace{4\pi \cdot 2^2}_{\text{Area of sphere}} = 16\pi \Rightarrow \text{Area}(S_2) = 16\pi - \text{Area}(S_1) = 12\pi$$

$\Rightarrow$ The areas of S_1 and S_2 are $\boxed{4\pi \text{ and } 12\pi}$